

Generalized quark number susceptibilities at finite chemical potential from fugacity expansion on the lattice

arXiv:1409.4672 [hep-lat], arXiv:1411.4143 [hep-lat], arXiv:1411.5133 [hep-lat]

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Lunch Time Talk
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Outline

Motivation

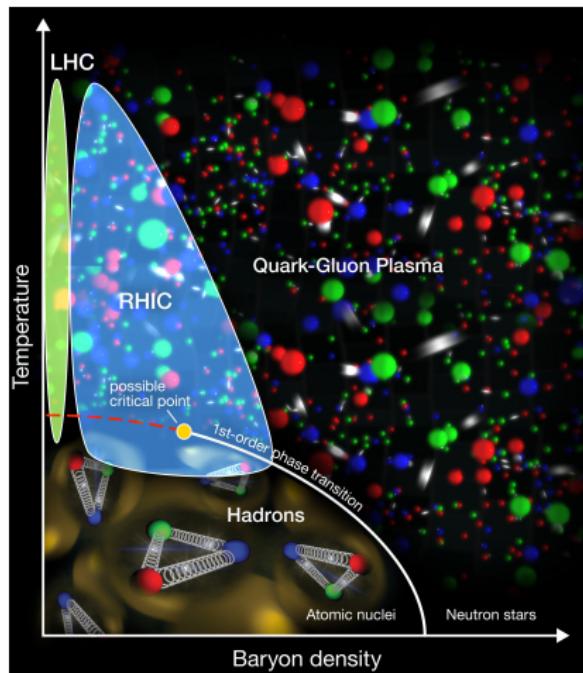
The fugacity expansion

Results for Wilson and staggered fermions

Canonical determinants from ChPT

Summary

The QCD phasediagram



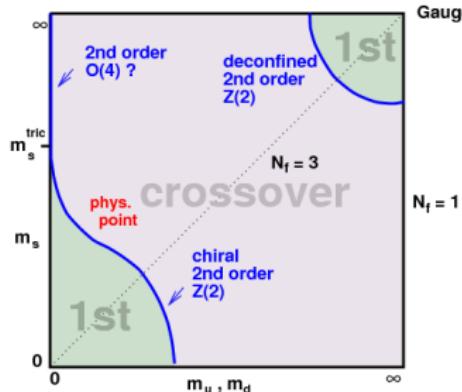
- ▶ Nature: Big bang, compact stars.
- ▶ Experiment: Heavy-ion collisions @ RHIC, LHC, FAIR.
- ▶ Recent experiments at RHIC: Evidence for 1st order transition.
[STAR Collaboration, \(2014\), \[arXiv:1401.3043 \[nucl-ex\]\].](#)
- ▶ Lack of good theoretical understanding.

<http://www.bnl.gov/newsroom/news.php?a=24473>

Exploring the phasediagram with lattice QCD

What we know - 1

- ▶ For the temperature driven phase transition we have lattice QCD.
- ▶ Columbia plot:

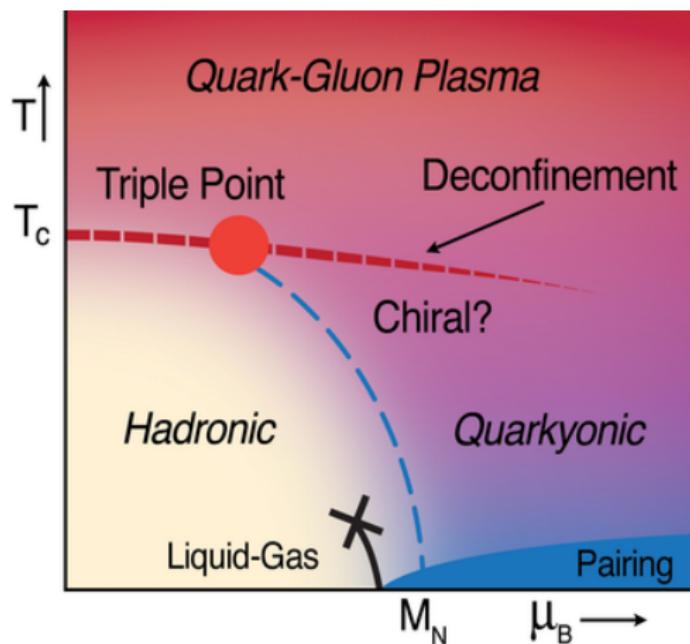


P. de Forcrand, et. al., PoS LATTICE2011 (2011) 189, arXiv:1201.2769 [hep-lat]

- ▶ Physical point: Crossover @ $T \approx 155\text{ MeV}$
- ▶ For reviews on what we know, see PoS LATTICExxxx.
- ▶ For reviews on what we don't know, see also PoS LATTICExxxx.

Exploring the phasediagram with lattice QCD

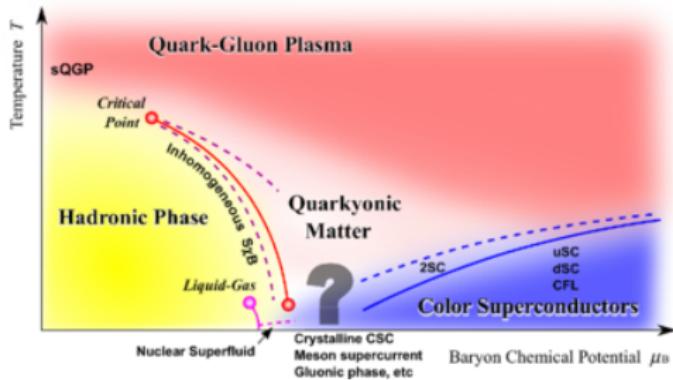
What we know - 2



Google image search "phase diagram qcd"

Exploring the phasediagram with lattice QCD

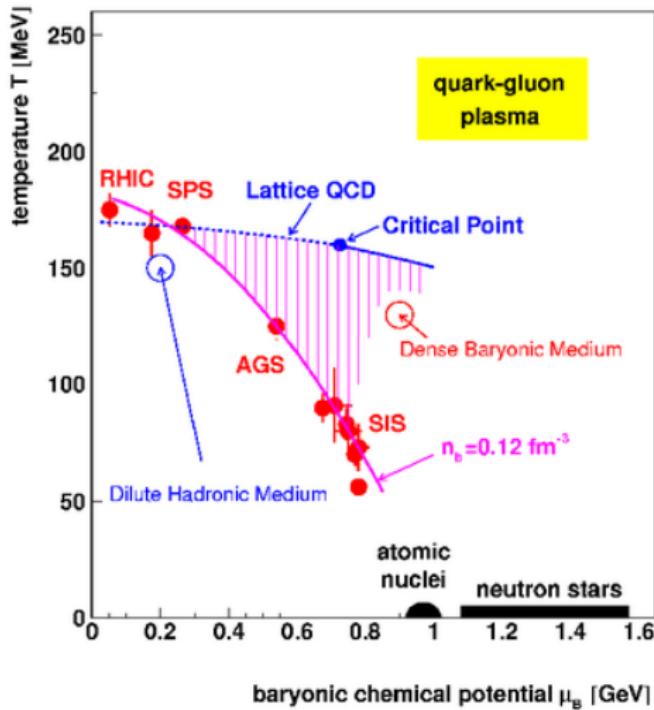
What we know - 2



[Google image search "phase diagram qcd"](#)

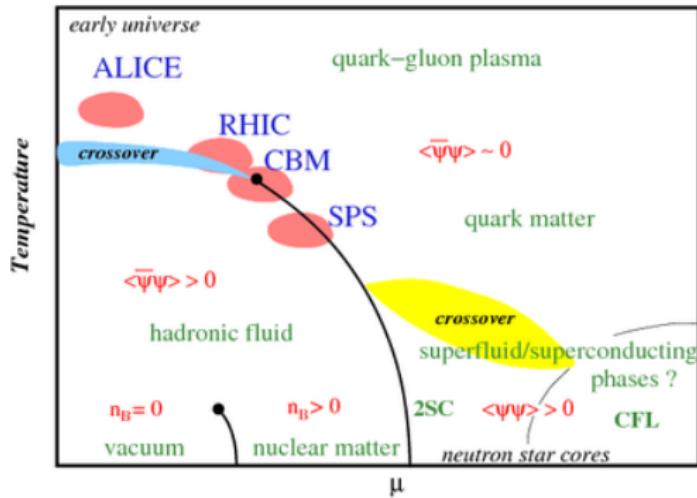
Exploring the phasediagram with lattice QCD

What we know - 2



Exploring the phasediagram with lattice QCD

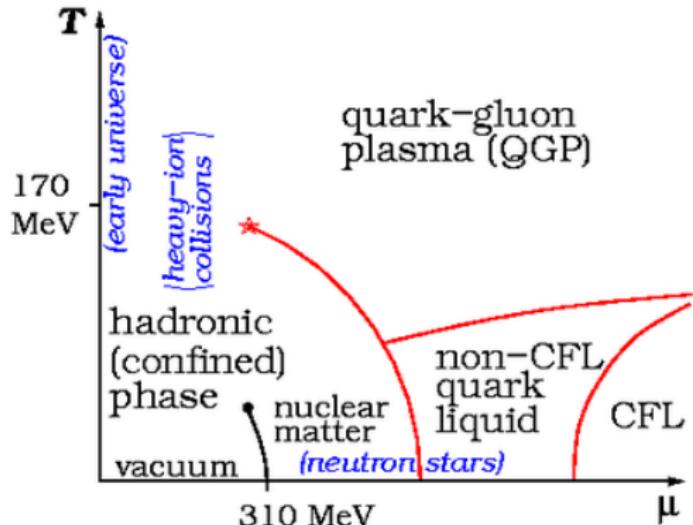
What we know - 2



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Exploring the phasediagram with lattice QCD

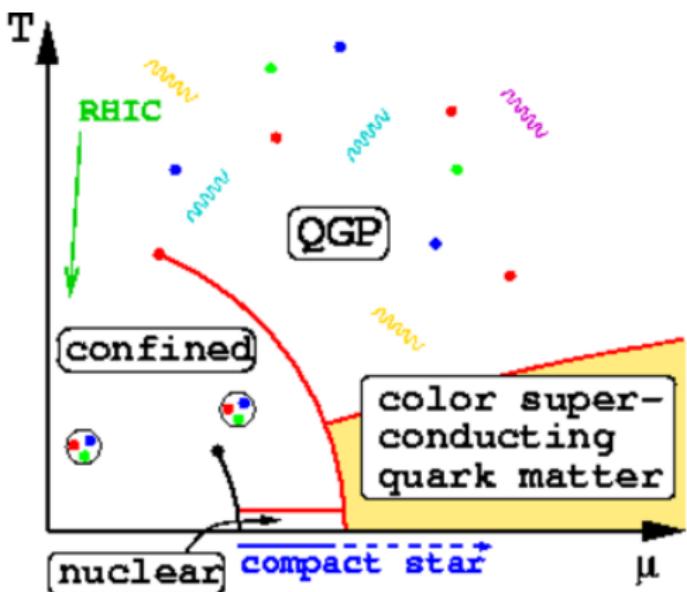
What we know - 2



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Exploring the phasediagram with lattice QCD

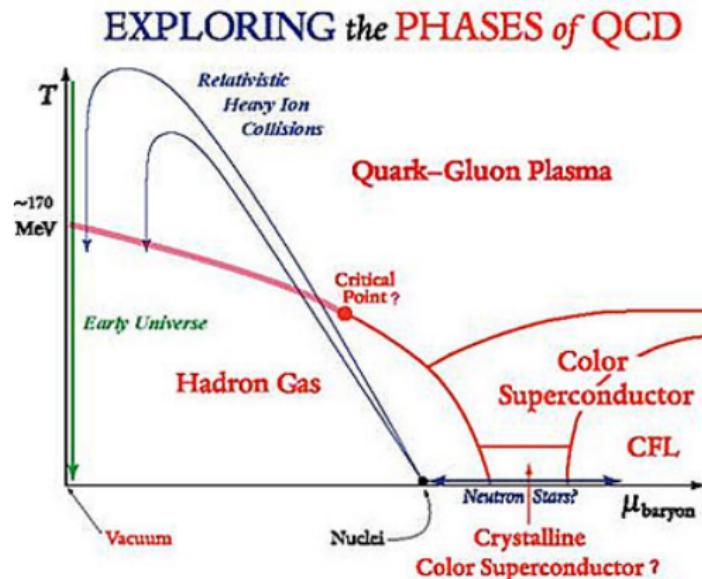
What we know - 2



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Exploring the phasediagram with lattice QCD

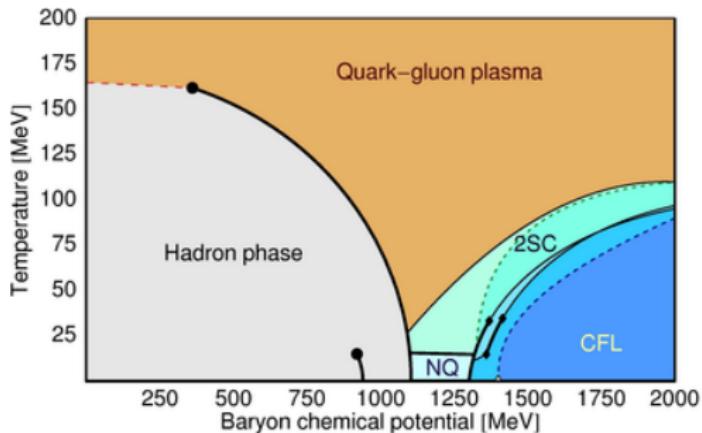
What we know - 2



Google image search "phase diagram qcd"

Exploring the phasediagram with lattice QCD

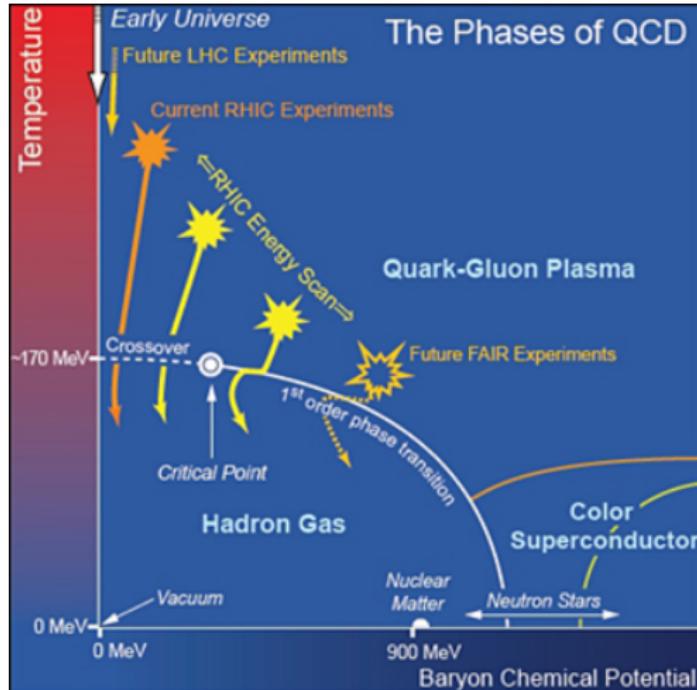
What we know - 2



[Google image search "phase diagram qcd"](#)

Exploring the phasediagram with lattice QCD

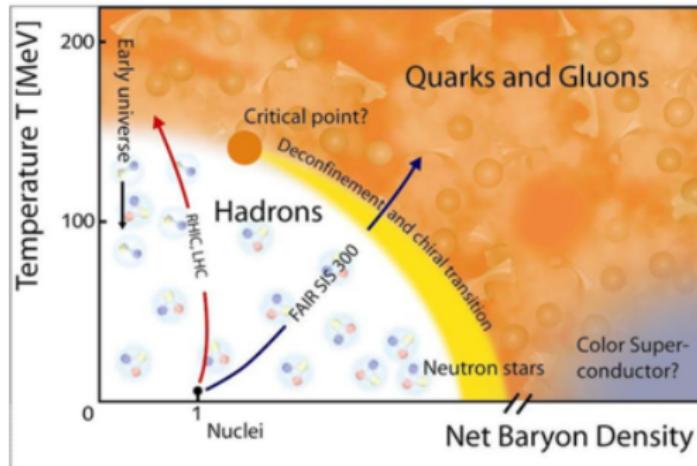
What we know - 2



Google image search "phase diagram qcd"

Exploring the phasediagram with lattice QCD

What we know - 2



Google image search "phase diagram qcd"

Exploring the phasediagram with lattice QCD

The sign problem - 1

- ▶ Major problem plagues us already a long time:
@ finite baryon chemical potential → **sign problem**.
- ▶ Short explanation: (D ... Dirac operator)

$$\begin{aligned} @ \mu = 0 : D &= \gamma_5 D^\dagger \gamma_5 \text{ (\gamma_5 hermiticity)} \\ \Rightarrow \det D &= \det \gamma_5 D^\dagger \gamma_5 = \det D^\dagger = (\det D)^* \\ \Rightarrow \det D &\in \mathbb{R} \end{aligned}$$

but:

$$\begin{aligned} @ \mu \neq 0 : D(\mu) &= \gamma_5 D(-\mu)^\dagger \gamma_5 \\ \Rightarrow \det D(\mu) &= \det \gamma_5 D(-\mu)^\dagger \gamma_5 = (\det D(-\mu))^* \\ &\neq (\det D(\mu))^* \\ \Rightarrow \det D(\mu) &\in \mathbb{C} \end{aligned}$$

- ▶ $\det D(\mu)$ cannot be used as a probability anymore!

Exploring the phasediagram with lattice QCD

The sign problem - 2

Is there a way out?

- ▶ Different approaches: Reweighting, Taylor expansion,
- ▶ Here: **Fugacity expansion.**
- ▶ Related studies:
A. Alexandru, M. Faber, I. Horvath, K.-F. Liu, Phys. Rev. D 72 (2005) 114513, [arXiv:hep-lat/0507020];
J. Danzer, C. Gattringer, Phys. Rev. D 86 (2012) 014502, [arXiv:1204.1020 [hep-lat]].
- ▶ Other methods:
Density of states, dual variables, complex Langevin.
- ▶ Recent reviews for $\mu \neq 0$:
D. Sexty, PoS LATTICE 2014 (2014), [arXiv:1410.8813 [hep-lat]];
C. Gattringer, PoS LATTICE 2013 (2013) 002, [arXiv:1401.7788 [hep-lat]];
G. Aarts, PoS LATTICE 2012 (2012) 017, [arXiv:1302.3028 [hep-lat]];
L. Levkova, PoS LATTICE 2011 (2011) 011, [arXiv:1201.1516 [hep-lat]].

The fugacity expansion – Motivation

- ▶ Fugacity expansion different from regular Taylor expansion (Laurent- v.s. Taylor-series & **finite sum for finite V**).
- ▶ Recently it was shown that it can have better convergence properties than a Taylor expansion (Z_3 model).
- ▶ OOH: **Numerically hard** (calculation of expansion coefficients).
- ▶ OTOH: Interesting observables easily accessible
⇒ **generalized quark susceptibilities**.

B. Friman, F. Karsch, K. Redlich, V. Skokov, Eur. Phys. J. C **71** (2011) 1694, [arXiv:1103.3511 [hep-ph]];
A. Bazavov, *et al.*, Phys. Rev. D **86** (2012) 034509, [arXiv:1203.0784 [hep-lat]];
A. Bazavov, *et al.*, Phys. Rev. D **88** (2013), 094021, [arXiv:1309.2317 [hep-lat]];
S. Borsanyi, *et al.*, Phys. Rev. Lett. **111** (2013) 062005, [arXiv:1305.5161 [hep-lat]];
S. Borsanyi, *et al.*, Phys. Rev. Lett. **113** (2014) 052301, [arXiv:1403.4576 [hep-lat]].

The fugacity expansion - 1

The **grand canonical determinant** with **real chemical potential** μ can be written as the **fugacity series** (exact for $q_{\text{cut}} = 6 V$)

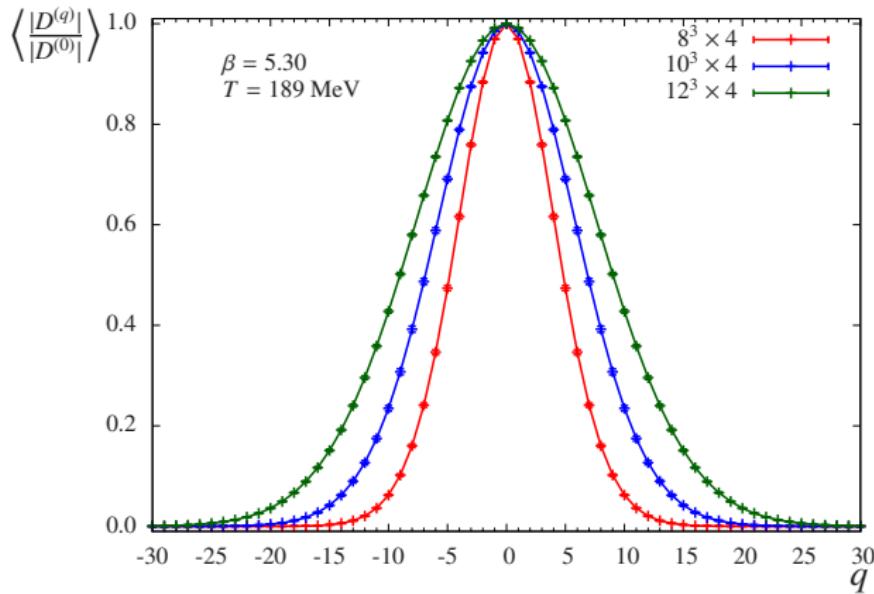
$$\det[D(\mu)] = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu\beta q} D^{(q)}.$$

$D^{(q)}$: **Canonical determinants** with net quark number q ,

$$D^{(q)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-iq\phi} \det[D(\mu\beta = i\phi)].$$

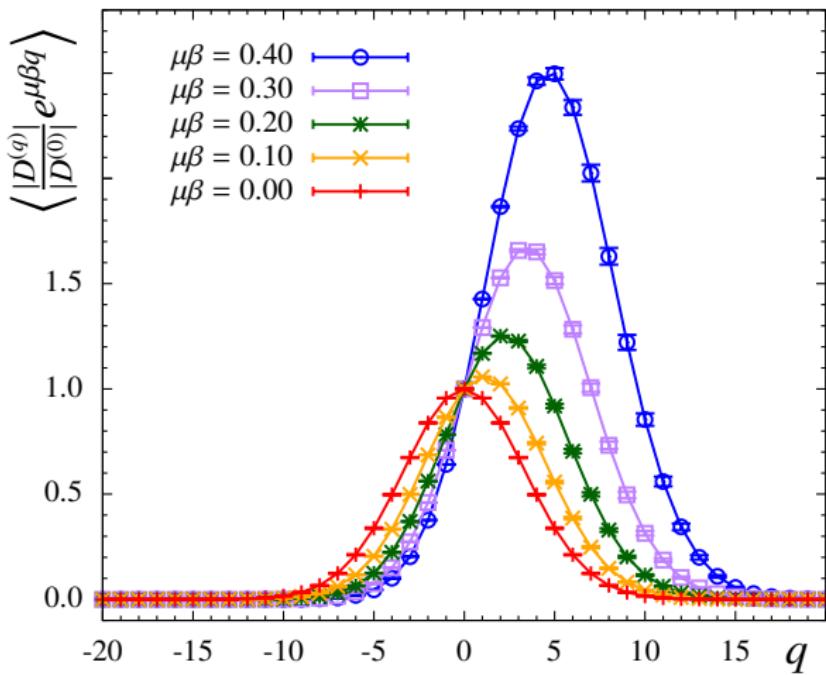
Fourier integral is done numerically $\Rightarrow q_{\text{cut}} \ll 6 V$. Usually $q_{\text{cut}} = O(10 - 100)$, depends on T , V , m_q ,

The fugacity expansion - 2



Dependence on q roughly linear with the lattice volume.

The fugacity expansion - 3



$$\det[D(\mu)] = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu\beta q} D^{(q)}$$

Technical details

Important to have $D^{(q)}$ at high precision.



Calculation of $\det[D(\mu\beta = i\phi)]$ for many values of ϕ .



Expensive (and limited by memory)!

Dimensional reduction: [J. Danzer, C. Gattringer, Phys. Rev. D 78 \(2008\) 114506 \[arXiv:0809.2736 \[hep-lat\]\].](#)

Use a domain decomposition of the Dirac operator $D(\mu)$ to obtain

$$\det[D(\mu)] = A W(\mu),$$

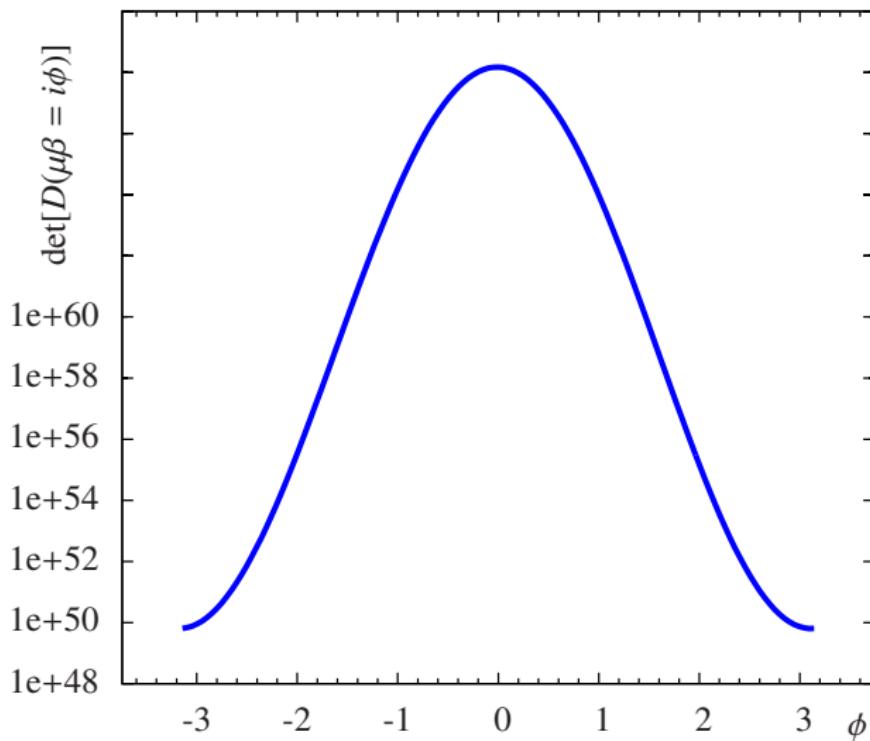
with a μ -independent factor A and

$$W(\mu) = \det[K_0 - e^{\mu\beta} K - e^{-\mu\beta} K^\dagger].$$

K_0, K are dense matrices living on a single time slice
($\dim K = N_s^3 \times 1 \times 3 \times 4$).

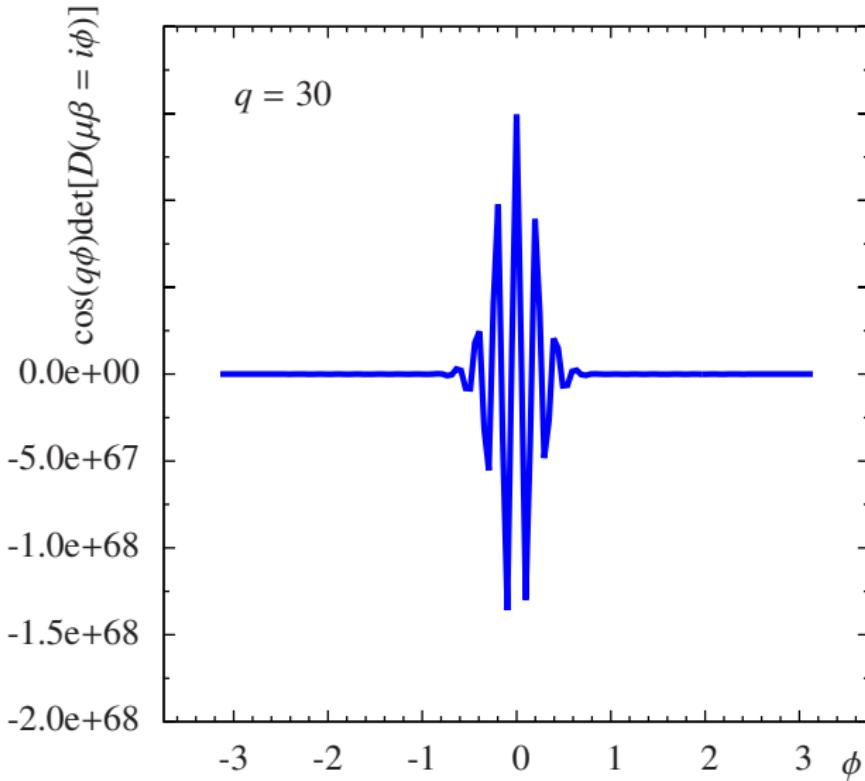
Done for: **Wilson**, Clover, **staggered** fermions.

Numerical integration: What do we integrate? - 1



$$D^{(q)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-iq\phi} \det[D(\mu\beta = i\phi)].$$

Numerical integration: What do we integrate? - 2



$$D^{(q)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \cos(q\phi) \det[D(\mu\beta = i\phi)] + \dots$$

Observables related to quark number - 1

Grand canonical partition sum written using fugacity series:

$$\begin{aligned} Z_\mu &= \int D[U] e^{-S_g[U]} \det[D(\mu)]^2 \\ &= \int D[U] e^{-S_g[U]} \left(\sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu \beta q} D^{(q)} \right)^2. \end{aligned}$$

Observables related to quark numbers are **derivatives w.r.t μ** , i.e.,

$$\chi_n^q \propto \frac{\partial^n \ln Z_\mu}{\partial (\mu \beta)^n},$$

and they take a **simple form in the fugacity approach**.

Observables related to quark number - 2

Moments of $D^{(q)}$:

$$M^n = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu \beta q} q^n \frac{D^{(q)}}{\det[D(\mu = 0)]}.$$

Quark number density (Wilson fermions):

$$\frac{\chi_1^q}{T^3} = \frac{n_q}{T^3} = 2 \frac{\beta^3}{V} \frac{\langle M^0 M^1 \rangle_0}{\langle (M^0)^2 \rangle_0}.$$

Quark number susceptibility (Wilson fermions):

$$\frac{\chi_2^q}{T^2} = 2 \frac{\beta^3}{V} \left[\frac{\langle (M^1)^2 \rangle_0 + \langle M^0 M^2 \rangle_0}{\langle (M^0)^2 \rangle_0} - 2 \left(\frac{\langle M^0 M^1 \rangle_0}{\langle (M^0)^2 \rangle_0} \right)^2 \right].$$

+ higher derivatives (3rd and 4th) and ratios.

$\langle \dots \rangle_0$: expectation value evaluated on configurations with $\mu = 0$.

Full QCD Results - Lattice parameters

Two flavor degenerate **Wilson fermions** & Wilson gauge action:

- ▶ Lattices $N_s^3 \times N_t$: $8^3 \times 4$, $10^3 \times 4$, $12^3 \times 4$, $\mathbf{12^3 \times 6}$ ($\beta = N_t = 1/T$)
- ▶ Inverse coupling: $5.00 \leq \frac{6}{g^2} \leq 5.70$
- ▶ Lattice spacing: $0.34 \text{ fm} \geq a \geq 0.10 \text{ fm}$
- ▶ Temperature: $100 \text{ MeV} \leq T \leq 500 \text{ MeV}$
- ▶ $\kappa = 0.158, 0.160, \mathbf{0.162}$ (pion mass: $M_\pi \leq 900 \text{ MeV}$)
- ▶ 50-100 configurations per parameter (for $12^3 \times 6$ lattices)

Two flavor degenerate **staggered fermions** & Wilson gauge action:

- ▶ Lattices $N_s^3 \times N_t$: $8^3 \times 4$, $\mathbf{16^3 \times 6}$, $16^3 \times 8$ ($\beta = N_t = 1/T$)
- ▶ Inverse coupling: $5.30 \leq \frac{6}{g^2} \leq 6.00$
- ▶ $m = \mathbf{0.10}, 0.05, 0.01$
- ▶ 100-150 configurations per parameter (for $16^3 \times 6$ lattices)

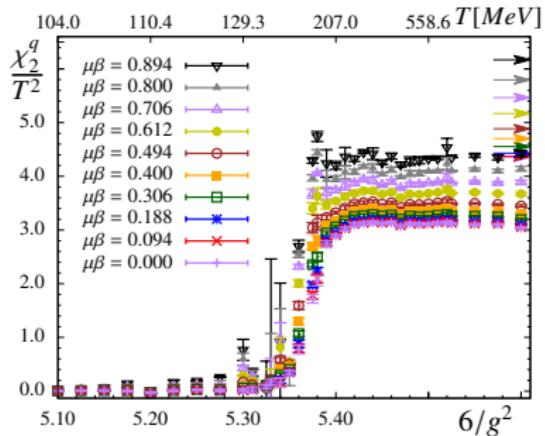
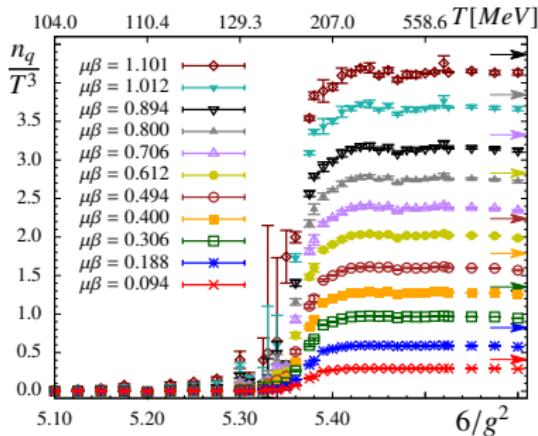
Configurations generated in-house using MILC code.

(<http://www.physics.utah.edu/~detar/milc/>)

Wilson: Quark number density & susceptibility

C. Gattringer, HPS, arXiv:1411.5133 [hep-lat].

($12^3 \times 6$, $\kappa = 0.162$)

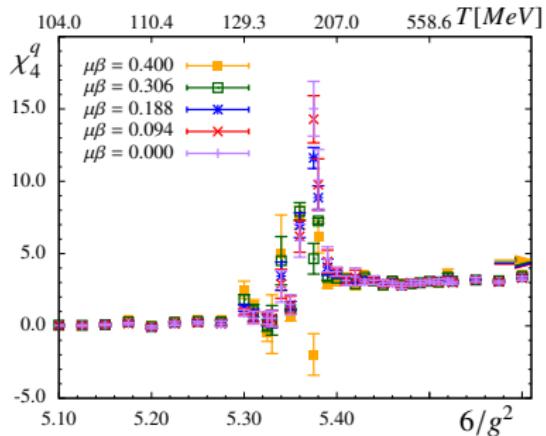
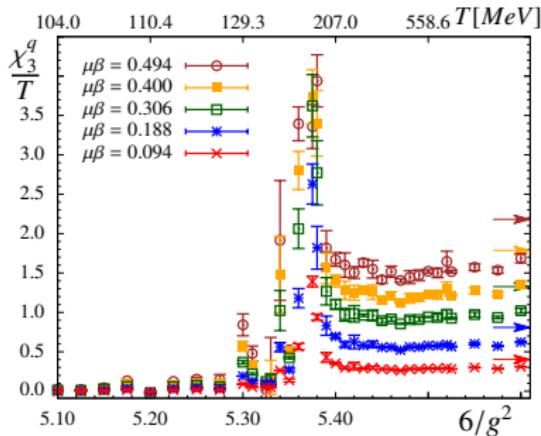


Quark number density (l.h.s.) and susceptibility (r.h.s.) as a function of $\frac{6}{g^2}$.

Wilson: Higher susceptibilities

C. Gattringer, HPS, arXiv:1411.5133 [hep-lat].

($12^3 \times 6$, $\kappa = 0.162$)

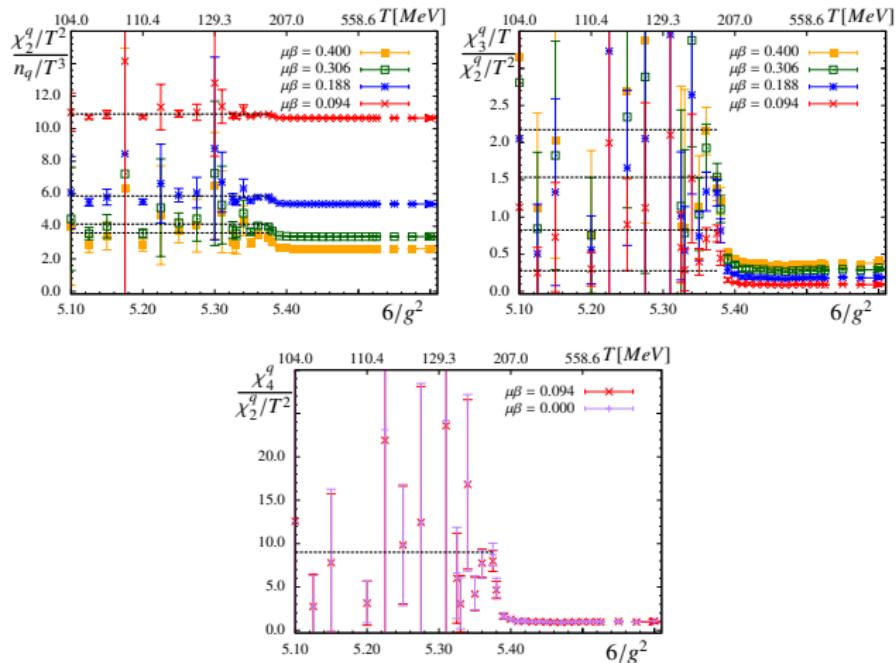


3rd derivative (l.h.s.) and 4th derivative (r.h.s.) as a function of $\frac{6}{g^2}$.

Wilson: Ratios of derivatives - 1

C. Gattringer, HPS, arXiv:1411.5133 [hep-lat].

$(12^3 \times 6, \kappa = 0.162)$



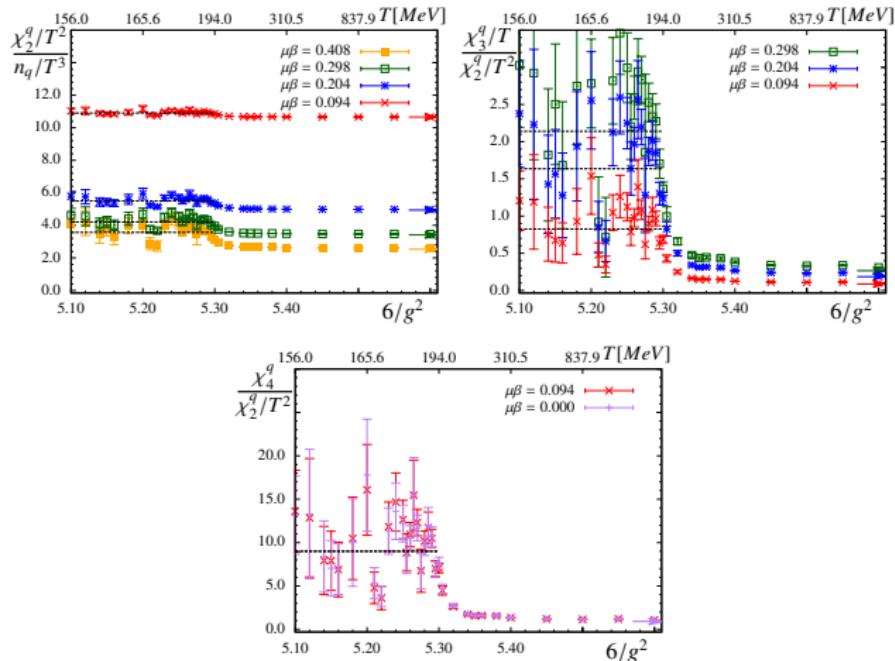
HRG (dashed lines):

$$\frac{\chi_2^q/T^2}{n_q/T^3} = 3 \operatorname{sech}(3\mu\beta), \quad \frac{\chi_3^q/T}{\chi_2^q/T^2} = 3 \tanh(3\mu\beta), \quad \frac{\chi_4^q}{\chi_2^q/T^2} = 9.$$

Wilson: Ratios of derivatives - 2

C. Gattringer, HPS, arXiv:1411.5133 [hep-lat].

$$(8^3 \times 4, \kappa = 0.158)$$



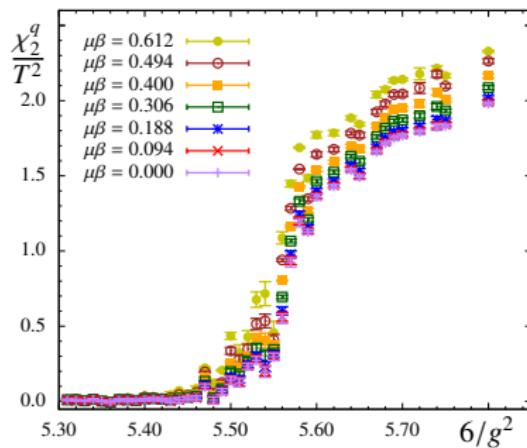
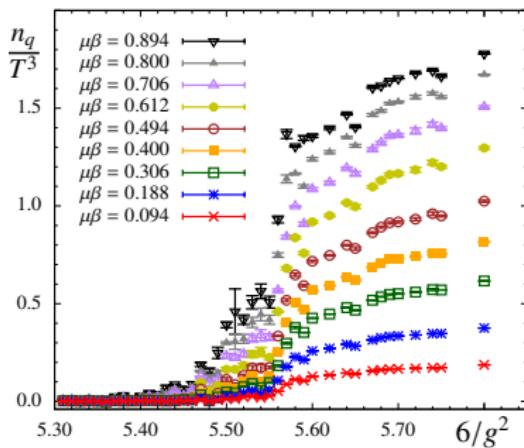
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Staggered: Quark number density & susceptibility

HPS, C. Gattringer, PoS Lattice 2014, [arXiv:1409.4672 [hep-lat]].

$(16^3 \times 6, m = 0.1)$

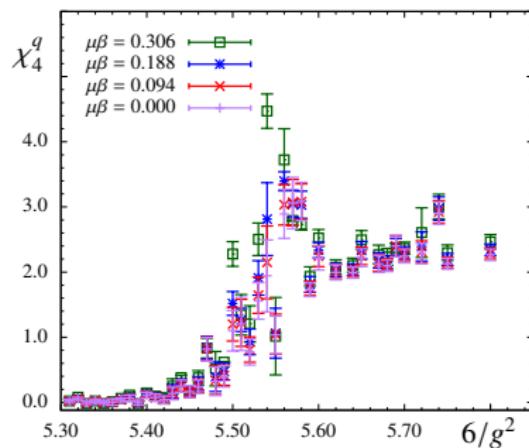
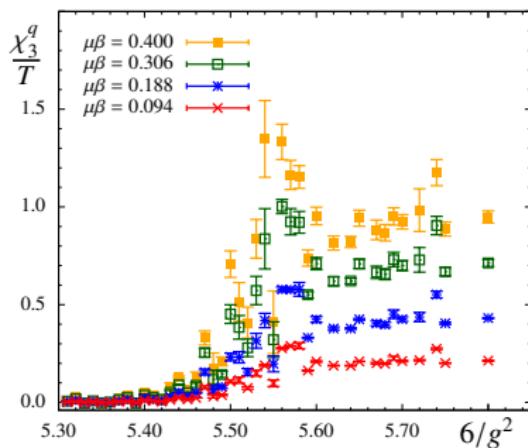


Quark number density (l.h.s.) and susceptibility (r.h.s.) as a function of $\frac{6}{g^2}$.

Staggered: Higher susceptibilities

HPS, C. Gattringer, PoS Lattice 2014, [arXiv:1409.4672 [hep-lat]].

$(16^3 \times 6, m = 0.1)$

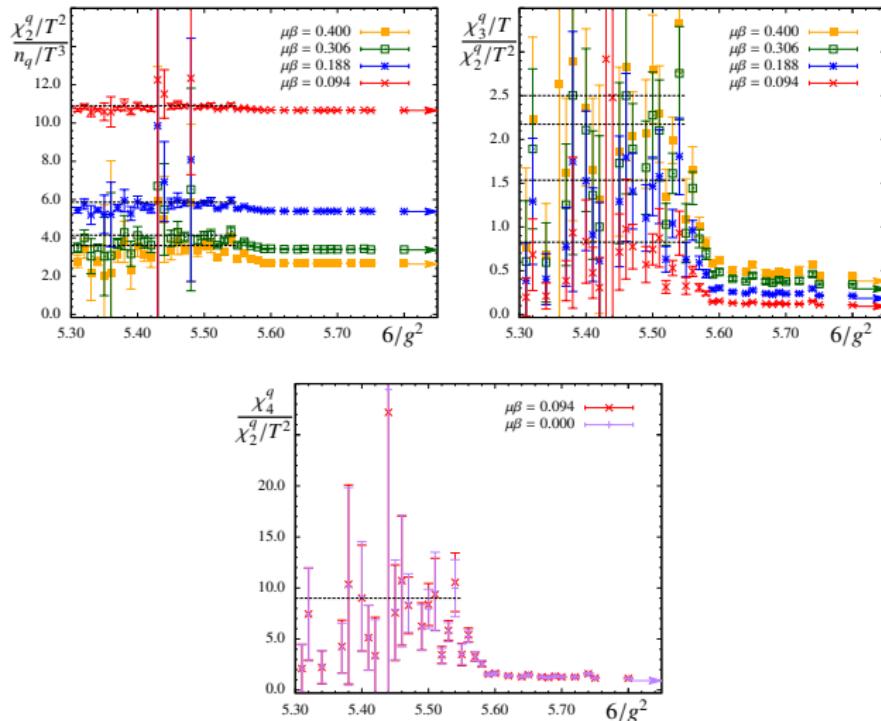


3rd derivative (l.h.s.) and 4th derivative (r.h.s.) as a function of $\frac{6}{g^2}$.

Staggered: Ratios of derivatives

HPS, C. Gattringer, PoS Lattice 2014, [arXiv:1409.4672 [hep-lat]].

($16^3 \times 6$, $m = 0.1$)



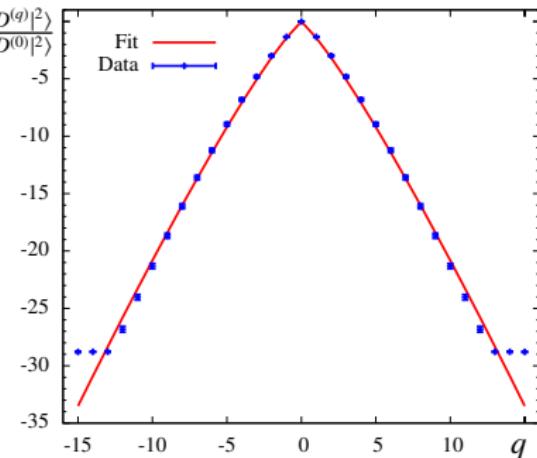
Note: HRG and free results are the same as in Wilson case.

$D^{(q)}$ from ChPT

A. Alexandru, C. Gattringer, HPS, K. Splittorff, J. Verbaarschot, arXiv:1411.4143 [hep-lat].

$$\frac{\langle |D^{(q)}|^2 \rangle}{\langle |D^{(0)}|^2 \rangle} = \frac{I_q \left(\frac{1}{\pi^2} A^3 (m_\pi \beta)^2 K_2(m_\pi \beta) \right)}{I_0 \left(\frac{1}{\pi^2} A^3 (m_\pi \beta)^2 K_2(m_\pi \beta) \right)}.$$

$A = \frac{Ns}{Nt}$... Aspect ratio
 I_q, K_2 ... Modified Bessel functions



Fit in $m_\pi \beta$ to $12^3 \times 6$, $T = 100$ MeV data.

Summary:

- ▶ Exploratory study of the Fugacity expansion.
- ▶ Different approach: (finite) Laurent series in $e^{\mu\beta}$.
- ▶ Observables related to $n_q \Rightarrow$ moments of $D^{(q)}$.
- ▶ Generalized susceptibilities can be calculated reliably at small μ .
- ▶ Fugacity approach can be extended to physically relevant V .
- ▶ Ratios: Interesting, very robust (cf. Wilson/staggered).

Outlook:

- ▶ Comparison of full QCD results with other expansion techniques.
- ▶ Study at smaller quark masses (need larger V and smaller a).

Thank you for your attention!